Communication and Contacts in Massively Interconnected Systems
Part 2: Connectivity Functions and Differential Equation of Connectedness

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Abstract

Because of its empirical nature the Rent's Rule is not able to capture the essence of connectivity in many cases of practical importance. To solve these problems of capturing the wide variety of cases of connectivity there is a need for more powerful spatial and temporal models of contacts and interaction between the elements of systems.

This paper discusses the methods of the theory of massively interconnected systems, developed in the 70s and 80s of the last century, as applied to one-dimensional systems of connectivity.

The concept of one-dimensional functions of connectivity is introduced and the differential equation with respect to these functions is derived.

This differential equation has wide practical applications in organizational science and electronics. Particularly the well-known Allen Curve is a specific solution of the differential equation of connectivity. The Rent's Rule itself can be derived from one-dimensional differential equation of connectivity.

Introduction

Numerous practical application of the Rent's Rule over the last 40-50 years have shown that its use is effective for estimating the parameters of microelectronic products, but at the same time it became clear that this rule has a limited scope and in many practical cases is not able adequately represent connectivity in complex systems [1, 2].

In particular the Rent's Rule is not suitable for the description of connectivity in spatially distributed systems with mass contacts and communications, such as social networks, organizations, project teams, and others.

The main reason for this is that generally the Rent's Rule in no way related to a particular coordinate system, while an arbitrary massively interconnected system has variable connectivity parameters and operates in real time and space.

The history of science shows that an adequate quantitative description of such systems, whose parameters vary in time and space, is usually reduced to the use of differential equations.

In this respect systems with mass communications and interactions between their elements are not particularly different from other systems with variable parameters and the natural mathematical means for their fundamental description are also differential equations with partial derivatives.

It is on the basis of such an approach the theory of massively interconnected systems is built and which in its general form is presented in [2, 3].

Some aspects of this theory related to the derivation and applications of differential equation of connectivity are presented in English in [4, 5, 6].

This paper is devoted to the presentation of a simplified one-dimensional version of this theory to illustrate its basic ideas and potentialities.

The main methodological shortcoming of the Rent's Rule as a model of connectivity in large systems

This drawback is that the Rent's Rule is a model of connectivity at the level of the group of elements and not at the level of the elements themselves.

The history of quantitative science clearly shows that it is much better to first build an adequate model at the level of elements of a system, then by generalization of a different nature to build a model of the system itself.

For example, in the theory of electricity Coulomb's law is a model of the electric field at the elementary level of the charge. Then, during the development of the methods of classical electrodynamics the Coulomb's law is organically included in the Maxwell's equations.

The same can be said of Newton's law of universal gravitation.

As a third example can be considered the theory of reliability of systems, where the reliability models of elements are also serve as the basis for generalizations and probabilistic description of the reliability of systems themselves.

Similar to electric charge and gravitational mass, which correspondingly generate the electric field and the gravitational field, the elements of large massively interconnected systems create a field of connectivity around, which, depending on the circumstances may be static or dynamic.

Similar the electric field or gravitational field, connectedness can also be described by differential equations with respect to connectedness functions of the elements of large systems.

The main characteristic of connectedness in large systems

The basis of each case of a quantitative description of a phenomenon is the distinctive features that are inherent in this specific phenomenon only.

Such a feature of massively interconnected systems is the weakening of connectedness of separate parts of the system with the increasing distance between them. In other words, the greater the distance between the individual parts of the system, the weaker their connectedness.

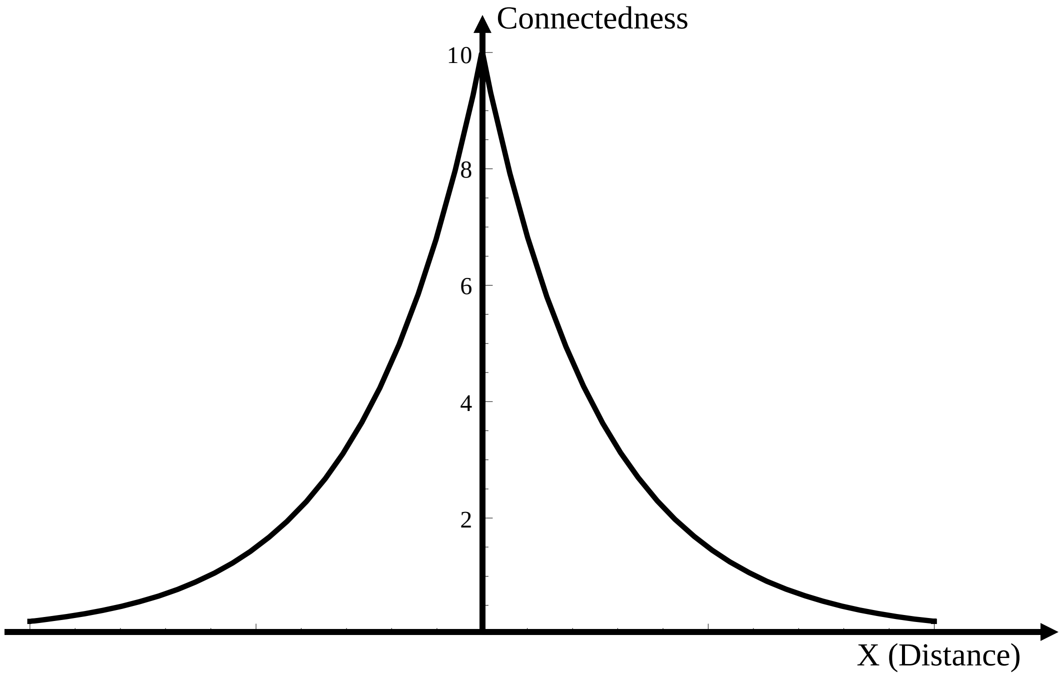

Fig.1 Weakening of connectedness with the increasing distance from the element (one-dimensional case)

In social systems, it simply means that on average, each person has contacts mainly with its immediate surroundings. In quantitative terms, this means that the number of contacts of a person is a decreasing function of distance from that person.

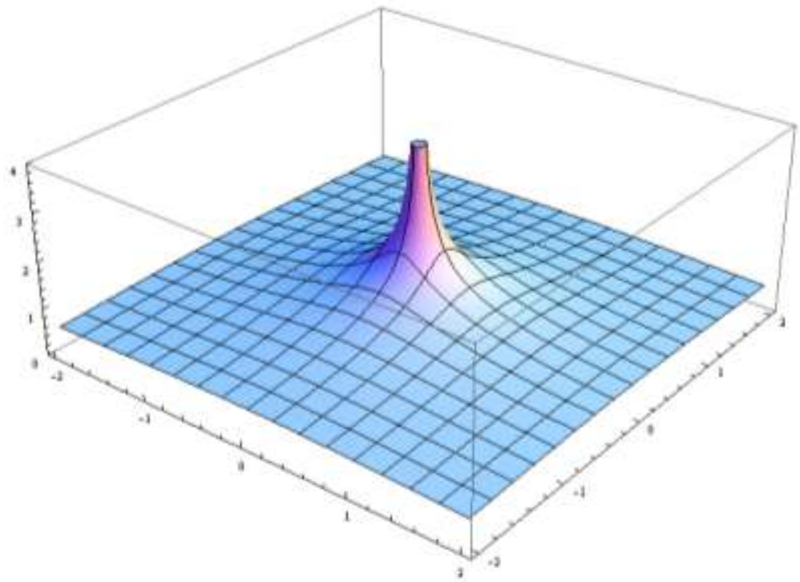

Fig.2 Weakening of connectedness with the increasing distance from the element (two-dimensional case)

Graphical representations of this characteristic feature for one- and two-dimensional cases are given in Fig.1 and Fig.2

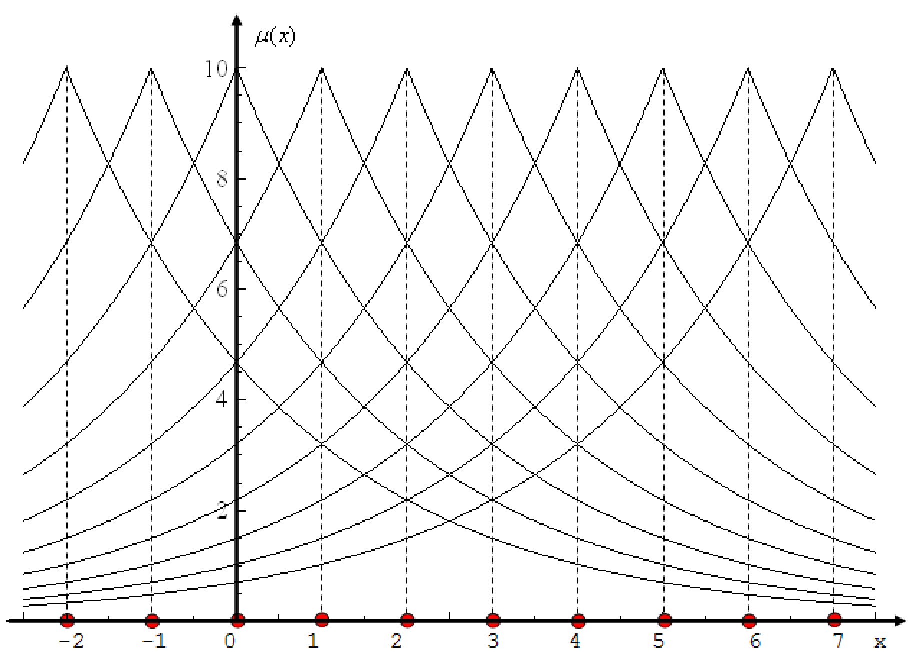

Fig.3 The chain of one-dimensional interconnected elements (red dots) along the x axis, each of which has the same connectedness function

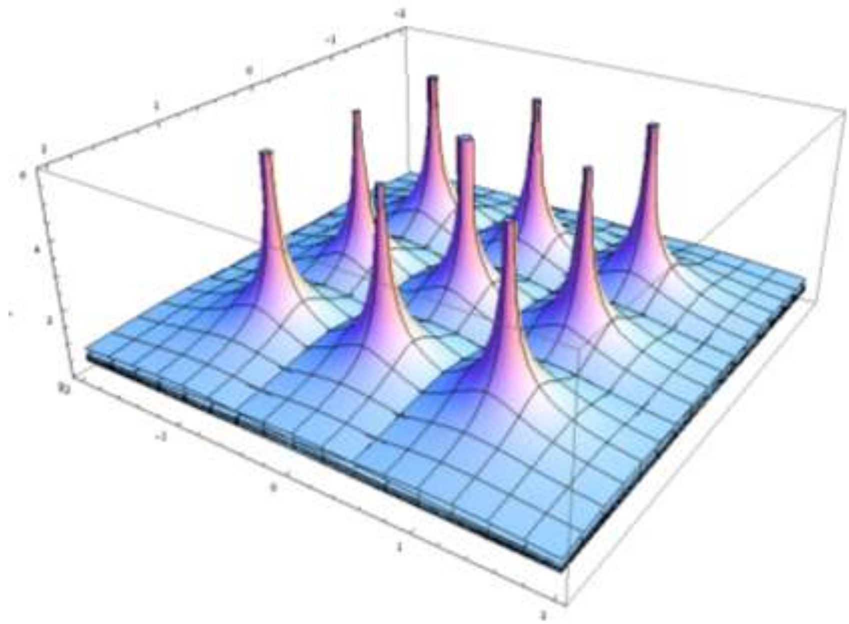

Fig.4 Group of interconnected elements each of which has the same two-dimensional connectedness function

Usually the mathematical modeling of a system requires replacing it with its idealized equivalent. In this sense in modeling of connectedness in large systems it is assumed that their elements have the same characteristics of connectedness.

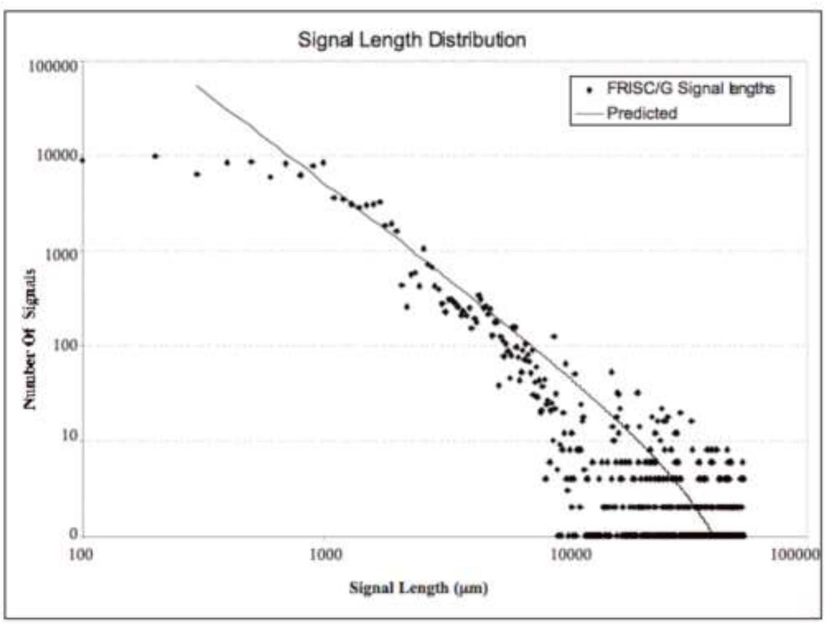

Fig.5 Typical wire length distribution density function in an electronic device

One-dimensional and two-dimensional groups of elements with identical connectedness characteristics are shown in Fig.3 and Fig.4.

The phenomenon of the weakening of connectedness with the increasing distance from the element in electronic devices means that the elements or groups of elements have wired signal connections mainly with their immediate neighbors. In other words, it means that the number of short signal wires in electronic systems is much higher than the number of long signal wires.

This is clearly seen in the figure, which shows a typical density function of the wire length in the electronic device [7].

The predominance of the number of short wires in electronic devices, the prevalence of contacts between immediate neighbors in the human systems, and many other similar examples in different systems indicate about the universality of this phenomenon. In turn, this phenomenon is a reflection of the extremity and rationality of related systems, regardless of the specific functions of the system.

Theory of Massively Interconnected Systems

The core of this theory is the differential equation of connectedness, which was derived in 1983 in order to quantify the wired connectivity between the big number of transistors and logic gates in semiconductor chips. The purpose of this research was the development of quantitative methods for estimating the physical parameters of chips and their feasibility analysis.

The basis of this general description of connectivity is the fact that in the real three-dimensional space between the elements of massively interconnected systems, there are numerous links that can undergo arbitrary changes over time. These links have a direction in space, which allows us to consider the set of links between elements of a system as a vector field, which can be described using differential equations.

Further studies showed that the differential equation of connectedness is applicable for the description of massively interconnected systems of different nature, as this equation was derived based on the general type of assumptions.

Consider the basic tenets of this theory for the special case of one-dimensional massively interconnected systems.

One-dimensional massively interconnected systems: Quantitative analysis

Let’s consider a one-dimensional massively interconnected system of elements that have identical connectedness characteristics (Fig.3). This is an idealization of real systems with large number of contacts and connections which can help to understand the essence of the theory of massively interconnected systems.

In such an idealized system elements with the same characteristics of connectedness, are located along the x axis with constant density or constant pitch.

The total number of contacts or connections of any element is a constant value equal to α that symmetrically distributed between the elements located before and after of that element.

Fig.3 shows a portion of the chain of elements arranged along the x axis, each of which has $\alpha = 20$ contacts per unit time (day, month), with the remaining elements. 10 of these contacts or connections are going to left and 10 of them - to right.

Thus, each element has its connectivity function which has two symmetrical branches to right and left, which reflect the nature of the distribution of contact of an element with the other elements.

For the analysis of the properties of connectedness functions, select the item located at the point "0" (Fig.6). Both branches of this function reflect the fact of reducing the number of contacts or connections with the items away from it. Element 0 has the same number of contacts with the immediate neighbors $n_{0,1}$ and $n_{0,-1}$. These values with the aid connectedness functions can be defined as follows.

$$n_{0,1} = n_{0,-1} = \mu(0) - \mu(1) = \mu(0) - \mu(-1)$$

The number of contacts of the element 0 with of the elements located in points 2 and -2 can be defined as

$$n_{0,2} = n_{0,-2} = \mu(1) - \mu(2) = \mu(-1) - \mu(-2)$$

Similarly the number of contacts between the elements 0 and 3 (or -3) will be
$n_{0,3} = n_{0,-3} = \mu(2) - \mu(3) = \mu(-2) - \mu(-3)$ and so on.

For our example shown in Fig.6 the average number of contacts of the element 0 with other elements are the following

$$n_{0,1} = n_{0,-1} = 10 - 6.84 = 3.16$$

$$n_{0,2} = n_{0,-2} = 6.84 - 4.68 = 2.16$$

$$n_{0,3} = n_{0,-3} = 4.68 - 3.2 = 1.48$$

$$n_{0,4} = n_{0,-4} = 3.2 - 2.187 = 1.01$$

$n_{0,5} = n_{0,-5} = 2.187 - 1.496 = 0.69$ and so on.

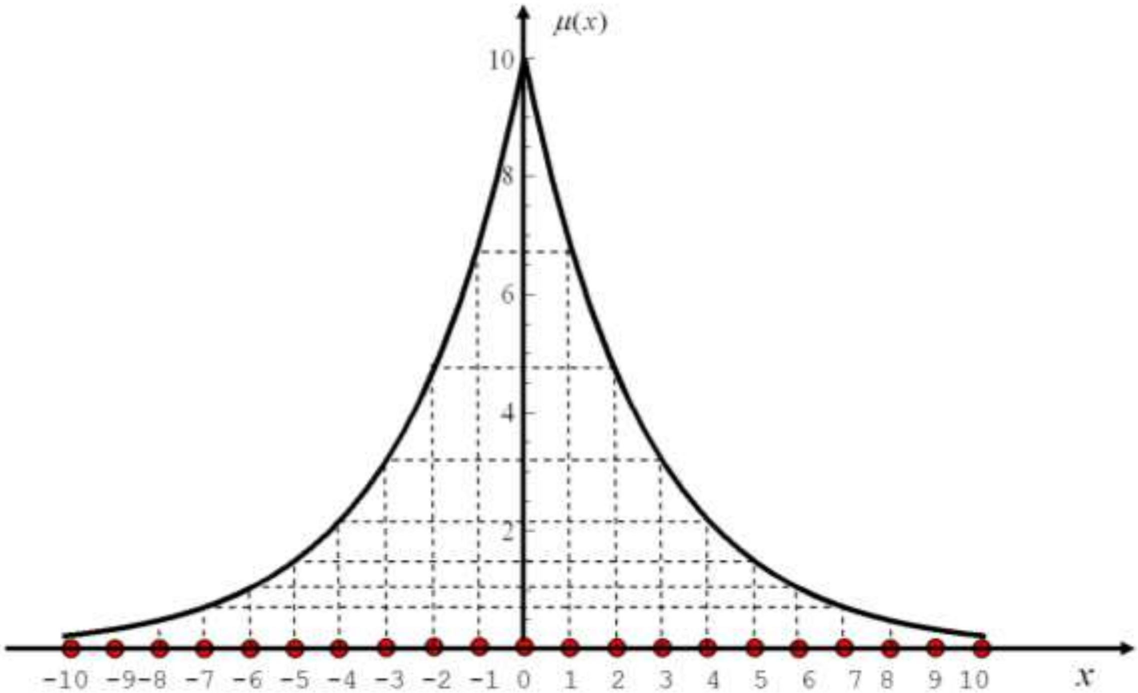

Fig.6 Two branches of the connectedness function for the element that is located in point "0"

Generalized description of connectedness functions

Let's now turn to a more generalized description of the properties of connectedness functions. Because of the symmetry of two branches of this function it is sufficient to analyze the properties of the right branch of it only (Fig.7).

The main property of connectedness function is that the value of this function at arbitrary point x indicates the number of contacts or communication of the element "0" with the elements located to the right of the point x.

Consider the process of reducing the number of links originated from the element at point "0".

Because element "0" has $\mu(x)$ connections with the elements located to the right of point x then it means that the same element has $\frac{\alpha}{2} - \mu(x)$ connections with the elements that locate in the interval [0, x].

Consider the behavior of the function $\mu(x)$ in the small interval Δx.
The number of connections $\Delta\mu$ that originated from element "0" and terminated in the small interval Δx can be calculated as

$$\Delta\mu = \mu(x) - \mu(x + \Delta x) \qquad (1)$$

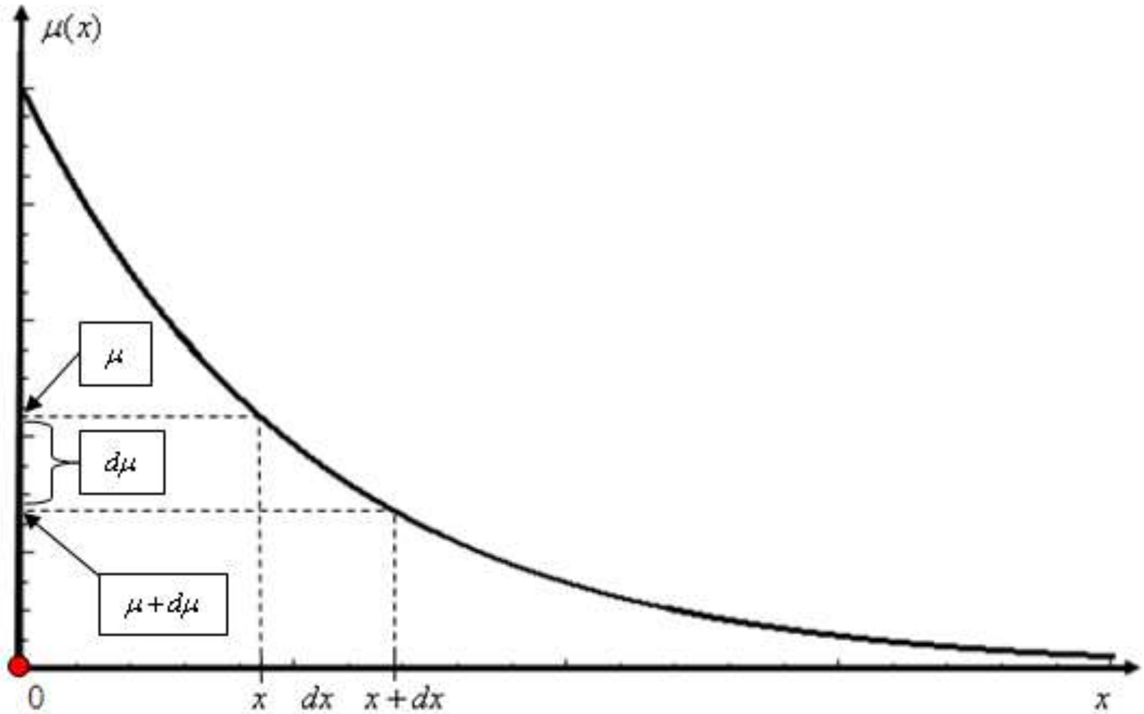

Fig.7 The right branch of the connectedness function for the element that is located in point "0"

Let's introduce the notion of the intensity $\gamma(x)$ of termination of connections originated from element "0" along with x axis. Because this $\gamma(x)$ function indicates the number of terminated contacts per unit length of the x axis then the product $\gamma(x)\Delta x$ will represent itself the probability of termination of the one of connections in the small interval Δx . Because the number of connections of element "0" in the section x is $\mu(x)$ then the total amount of terminated connections in the small interval Δx will be

$$\Delta\mu = \mu(x)\gamma(x)\Delta x \qquad (2)$$

Equalizing two values of $\Delta\mu$ from (1) and (2) we can have

$$\mu(x) - \mu(x + \Delta x) = \mu(x)\gamma(x)\Delta x \text{ or } \mu(x + \Delta x) - \mu(x) = -\mu(x)\gamma(x)\Delta x \qquad (3)$$

Then dividing the two sides of this equality by an amount Δx one can obtain

$$\frac{\mu(x + \Delta x) - \mu(x)}{\Delta x} = -\mu(x)\gamma(x) \qquad (4)$$

Then passing to the limits $\Delta x \to 0$ we obtain the differential equation of connectedness for one-dimensional case

$$\frac{d\mu(x)}{dx} = -\mu(x)\gamma(x) \quad (5)$$

This is the simplest form of the differential equation of connectedness [5, 6]. The solution of this equation with the boundary condition $\mu(0) = \frac{\alpha}{2}$ has the form

$$\mu(x) = \frac{\alpha}{2} Exp[-\int_0^x \gamma(y)]dy \quad (6)$$

In practice connection termination function $\gamma(x)$ depending on the circumstances can have infinite number of different mathematical forms. The simplest among them is the case when $\gamma(x) = Constant = \gamma$ which as a specific case leads to the well known Allen Curve.

Of particular interest to the electronics is a special case of hyperbolic functions $\gamma(x)$ of the intensity of termination of connections or links which makes it possible to derive the Rent's Rule analytically from the same differential equation of connectedness [5].

The distribution function of the distance between interacting elements

Since by definition $\mu(x)$ is the number of connections in the section "x" that have been originated from element "0", then the ratio $\mu(x)/\frac{\alpha}{2}$ will be the probability that the distance between the elements or the length of connections of that element will exceed "x".

Consequently we can define the connection length distribution function as

$$F(x) = 1 - \frac{2\mu(x)}{\alpha} \quad (7)$$

Solving this equation with respect to connectedness function $\mu(x)$ we obtain

$$\mu(x) = \frac{\alpha(1 - F(x))}{2} \quad (8)$$

Substituting this value of $\mu(x)$ into the differential equation of connectedness we obtain a new differential equation with respect to the connection length distribution function

$$\frac{dF(x)}{dx} = \gamma(x)(1 - F(x)) \quad (9)$$

The solution of this equation has the form

$$F(x) = 1 - Exp[-\int_0^x \gamma(y)]dy \quad (10)$$

Differentiation of this function gives us the density function of the connection length or the distance between interacting elements.

$$\varphi(x) = \frac{dF(x)}{dx} = \gamma(x)Exp[-\int_0^x \gamma(y)]dy \quad (11)$$

Allen Curve as a specific solution of one-dimensional differential equation of connectedness

As it been mentioned above the specific case $\gamma(x) = Constant = \gamma$ corresponds to the Allen Curve [8]. In this specific case the differential equation for one element that locates in point “0” has the form

$$\frac{d\mu(x)}{dx} = -\gamma\mu(x) \quad (12)$$

Substituting in (6) $\gamma(x) = \gamma$ we obtain the solution of this equation in the following form

$$\mu(x) = \frac{\alpha}{2} Exp(-\gamma x) \quad (13)$$

Corresponding differential equation for the connection length will have the form

$$\frac{dF(x)}{dx} = \gamma(1 - F(x)) \quad (14)$$

Connection length density function or the distance frequency between elements will have the form

$$\varphi(x) = \gamma Exp(-\gamma x) \quad (15)$$

This solution of the differential equation of connectedness represents itself the Allen Curve or the exponential drop of frequency of communication between elements as the distance between them increases.

Conclusions

1. Because of its empirical nature the Rent's Rule is not able to capture the essence of connectedness in many cases of practical importance. To solve these problems of capturing the wide variety of cases of connectedness there is a need for more powerful spatial and temporal models of contacts and interaction between the elements of systems.
2. In particular the Rent's Rule is not suitable for the description of connectedness in systems with mass contacts and communications, such as social networks, organizations, project teams, and others.
3. The main methodological shortcoming of the Rent's Rule is that it is a model of connectedness at the level of the group of elements and not at the level of the elements themselves.
4. The history of science shows that an adequate quantitative description of such systems, whose parameters vary in time and space, is usually reduced to the use of differential equations.
5. In this respect massively interconnected systems are not particularly different from other systems with variable parameters and the natural mathematical means for their fundamental description are also differential equations with partial derivatives.
6. The main characteristic of connectivity in large systems is the weakening connectedness of separate parts of the system with the increasing distance between them. In other words, the greater the distance between the individual parts of the system, the weaker their connectedness.
7. The mathematical theory of connectedness of massively interconnected systems that has been developed between 1972 -1983 of the last century is applicable in many areas of knowledge including social sciences, organizational science, project and program management, and others.
8. The main results of the theory of massively interconnected systems can be illustrated using its one-dimensional simplified version.
9. The well-known Allen Curve is the specific solution of one-dimensional connectedness equation.
10. Rent's Rule can be derived from one-dimensional connectedness equation too.

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Dr. Pavel Barseghyan is a consultant in the field of quantitative project management, project data mining and organizational science. Has over 40 years' experience in academia, the electronics industry, the EDA industry and Project Management Research and tools development. During the period of 1999-2010 he was the Vice President of Research for Numetrics Management Systems. Prior to joining Numetrics, Dr. Barseghyan worked as an R&D manager at Infinite Technology Corp. in Texas. He was also a founder and the president of an EDA start-up company, DAN Technologies, Ltd. that focused on high-level chip design planning and RTL structural floor planning technologies. Before joining ITC, Dr. Barseghyan was head of the Electronic Design and CAD department at the State Engineering University of Armenia, focusing on development of the Theory of Massively Interconnected Systems and its applications to electronic design. During the period of 1975-1990, he was also a member of the University Educational Policy Commission for Electronic Design and CAD Direction in the Higher Education Ministry of the former USSR. Earlier in his career he was a senior researcher in Yerevan Research and Development Institute of Mathematical Machines (Armenia). He is an author of nine monographs and textbooks and more than 100 scientific articles in the area of quantitative project management, mathematical theory of human work, electronic design and EDA methodologies, and tools development. More than 10 Ph.D. degrees have been awarded under his supervision. Dr. Barseghyan holds an MS in Electrical Engineering (1967) and Ph.D. (1972) and Doctor of Technical Sciences (1990) in Computer Engineering from Yerevan Polytechnic Institute (Armenia). Pavel's publications can be found here: http://www.scribd.com/pbarseghyan and here: http://pavelbarseghyan.wordpress.com/.

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