The algorithm for generating an optimal investment portfolio

Gelrud Y. D.

Introduction

The term *investment portfolio* means a certain set of *securities* owned by a natural person or a legal entity which acts as a complete *management* object. The main problem that must be addressed when forming *securities holdings* is the task of distributing a certain amount of money on various alternative investments (such as shares, bonds, cash, etc.) by an *investor* so that their goals are achieved in the best way.

In the first place investor seeks to obtain maximum income by means of: benefit from a favorable shares exchange rate; dividends; receiving solid interest, etc. On the other hand, any capital investment is associated with not only the expectation of income, but also with the constant danger of loss. Therefore, it is essential to take risk into account when solving optimization problems of choice of the securities holdings. The meaning of the portfolio is to improve the investment environment, giving the aggregate of securities such investment characteristics that are unattainable from a position of individual securities and are only possible in their combination.

Income on portfolio investments are gross profit on the entire set of securities included in a particular portfolio taking risk into account. The problem of quantitative correspondence between profit and risk arises and it must be addressed promptly in order to continuously improve the structure of already built and building of new portfolios in accordance with the wishes of investors. Taking into account the investment qualities of securities we can generate a variety of portfolios, each of which will have its own balance between the existing risk acceptable to the owner of the portfolio, and the expected return (income) during a certain period of time. The ratio of these factors allows us to determine the type of portfolio securities.

One of the most effective methods of assessment in the preparation of the investment portfolio is modeling. Modeling allows one to obtain the required investment characteristics of a future portfolio depending on the prevailing market conditions in a short time.

1. Problem statement and a description of the methods used.

Consider the following optimization model.

Let the economic entity has financial assets in the amount of Q on the interval $[0, T]$. It is known that it can use those assets for the acquisition of n types of securities, limited to volumes $V_1, \ldots, V_n$. The initial cost of one unit of securities of type i is s_i, and the projected cost of one unit of securities of type i at time T is d_i. The task is to select such types of securities and their number in order to maximize profit gained after the sale of all types of securities purchased at time T. The problem of formation of portfolio securities may be formed as the following problem of integer linear programming:

$$x_i \in N \ \ x_i \leq V_i$$

$$\sum_{i=1}^{n} s_i x_i \le Q \qquad \text{(I)}$$

$$\sum_{i=1}^{n} (\alpha d_i - s_i) x_i \to \max$$

where α - the discount factor - the factor used to convert future earnings to the current price. Factor present value (discount factor) of cash flows beyond the planning period is calculated by the formula:

$\alpha = 1 / (1 + r)^T$, where

T - number of years;
r - the discount rate chosen .

This problem is the knapsack problem with one-dimensional constraints, and belongs to the so-called NP-hard problems, which are characterized by the exponential growth of computing with the growth of dimension of the problem (the algorithm of its solution for the case α = 1 is given in [3]).

One of the problems arising in the practical use of solution of the proposed problem is the reliability of the forecast value of the securities d_i (i = 1 ... n). If we know the distribution function of the random variables that determines the possible profit of each type of securities, then the portfolio is chosen so that it maximizes the expectation value of profit or minimises the risk of financial loss (standard deviation). The first similar model was proposed in the article "Portfolio Selection" by G. Markowitz, and it provides methods for constructing such portfolios under certain conditions. The main merit of G. Markowitz in this article is theoretic and probabilistic formalization of the concepts of "*yield*" and "*risk*". *The expected return on a portfolio* of securities is defined as the average value of the discount profit, and *risk* - as the variance (the sum of squared deviations of the possible values of the average expected yield).

G.Markovits initially considered d_i as discretely distributed random variables, that is variables which value are d_{ij} with probability p_{ij} (j = 1, ..., k). Let us introduce the notation

$D_i = \alpha \sum_{j=1}^{k} p_{ij} d_{ij} - s_i$ - discount expectation of profit per unit of securities of the form i.

Then the problem (I) was seen in two versions, and took the following form:

$$x_i \in N \quad x_i \le V_i$$

$$\sum_{i=1}^{n} s_i x_i \le Q \qquad (\mathrm{II}_1)$$

$$\sigma_D^2 = \sum_{i=1}^{n}\sum_{j=1}^{k} p_{ij}[(\alpha d_{ij} - s_i)x_i - M(\alpha d_i - s_i)x_i]^2 = \sum_{i=1}^{n}\sigma_{D_i}^2 \le R$$

$$\sum_{i=1}^{n} M(\alpha d_i - s_i)x_i = \sum_{i=1}^{n}(\alpha\sum_{j=1}^{k} p_{ij}d_{ij} - s_i)x_i = \sum_{i=1}^{n} D_i x_i \to \max$$

(maximization of the discount expectation of profit while limiting the risk of financial loss).

$$x_i \in N \quad x_i \le V_i$$

$$\sum_{i=1}^{n} s_i x_i \le Q \qquad (II_2)$$

$$\sum_{i=1}^{n}(\alpha\sum_{j=1}^{k} p_{ij}d_{ij} - s_i)x_i = \sum_{i=1}^{n} D_i x_i \ge D$$

$$\sigma_D^2 = \sum_{i=1}^{n}\sum_{j=1}^{k} p_j[(\alpha d_{ij} - s_i)x_i - M(\alpha d_i - s_i)x_i]^2 \to \min$$

(minimizing the risk of financial loss under restriction to discount expectation of profit, where D - the desired efficiency of the market as a whole).

Algorithms for the solution and the results for such a task are described in detail in [2]. It should only be noted that G.Markowitz as well as subsequent authors did not discount future earnings, although we can give a numerical illustration in which the results of the optimization without discounting and with discounting, introduced in this article, are different. Modification of these algorithms in connection with the introduction of the discount factor $\alpha \neq 1$ is quite obvious and therefore it is not introduced in this article.

G. Markowitz did not stop there and continued developing the basic principles of building a portfolio. These principles form the basis for many papers describing the relationship between risk and return. In the first half of the 1960s William Sharpe, a student of Markowitz, proposed the so-called single-factor model of the capital market, which was first to introduce the "alpha" and "beta" shares characteristics which have become famous afterwards.

Mathematical single-factor regression Sharpe's model is illustrated as following:

$$D_i = \alpha_i + \beta_i D + \varepsilon_i, \ i=1,...,n, \qquad (1)$$

where D - efficiency of the overall market. Implementation of this efficiency is calculated as the weighted average options of implementations of D_i with weights proportional to the volume of total capital investment market, placed in the i-th shares.

The parameters of the regression model (1) α_i and β_i are called alpha and beta factors, and their values for the shares of leading corporations and firms are regularly published in the press and passed through various communication channels.

The Sharpe's model (1) assumes that $D_1, ..., D_n$ are independent, that is cross-correlation of the efficiencies of different types of shares is due only to their relationship through the effectiveness of the market as a whole.

The elements of the covariance matrix W are defined through the beta-factors of the model (1):

$$W_{ij}=\beta_i\beta_j \sigma_D^2 , i,j= 1,...,n , i\neq j,$$

$$W_{ii}= \sigma_{D_i}^2 = \beta_i^2 \sigma_D^2 + \sigma_{\varepsilon_i}^2$$

Thus, the risk of investing in shares of i-type $\sigma_{D_i}^2$ consists of two risks: $\beta_i^2 \sigma_D^2$ is a risk due to random fluctuations of the effectiveness of capital investments into i-th shares because of random fluctuations of market overall efficiency (risk factor); $\sigma_{\varepsilon_i}^2$ is the risk caused by its own independent random fluctuations of effectiveness of i-th shares (non-factor risk).

We can show that the diversification of investment capital leads to an averaging of a factor (market) risk and a reduction of a non-factor (equity) risk.

Thus, beta-factor shows the contribution of each security into the risk of the market portfolio, that is why it is not surprising that the risk premium required by investors is proportional to the beta-factor.

Currently, in addition to the one-factor Sharpe's model, multi-factor models are used that can be illustrated as follows:

$$D_i=\alpha_i+\beta_{i1}F_1+...+\beta_{im}F_m, i=1,...,n, \qquad (2)$$

where $F_1, ..., F_m$ - factors affecting the change in the shares' effectiveness.

The main factors taken into account usually are: interest rates; the inflation rate; prices of basic raw materials (oil, gas, etc.); industry factors.

Developed countries widely use BARRA multifactor model, which contains 68 basic and industry factors [2].

In addition to traditional single-level multi-factor models, including BARRA model, the two-level multi-factor models can be used [4]. For instance, taking into account three factors (the effectiveness of the market in general D, efficiency of the industry D^I and of the region D^r), to which a considered enterprise issuing i-th shares belongs , the two-level model looks as follows:

$$D_i = a_i + b_{i1}D + b_{i2}D^I ... + b_{i3}D^r, \; i=1,...,n, \tag{3}$$

According to the two-level model (3) stochastic dependence of the efficiency D_i from the market is carried out not only directly, but also through the stochastic dependence D_i on the effectiveness of the industry and the region, to which a considered issuing enterprise belongs.

The possibility of such an approach is of particular importance for the stock market in the Russian Federation, where the influence of industries and regions on the financial and economic characteristics of the business is of the highest priority.

(3) leads to the following representation for the variance (risk in the Markowitz scheme) of the i-th issuer:

$$\sigma^2_{D_i} = \sigma^2_{\varepsilon_i} + b^2_{i1} s^2_D + \beta^2_{i2}\sigma^2_{D^I} + \beta^2_{i3}\sigma^2_{D^r},$$

where

$\sigma^2_{\varepsilon_i}$ is a component of risk due to independent fluctuations in the issuer's efficiency;

$\beta^2_{i2}\sigma^2_{D^I}$ is a component of risk due to industry's own fluctuations in the efficiency, which include the issuer;

$\beta^2_{i3}\sigma^2_{D^r}$ is a component of risk due to region's own fluctuations in the efficiency, to which the issuer belongs;

$\beta^2_{i1}\sigma^2_D$ is a component of risk due to fluctuations in the efficiency of the overall market.

2. Description of the proposed dynamic approach.

We have examined the optimization problem of investment portfolios in the statements modifying the classical Markowitz optimization scheme, including the cases of instability of optimization problems. Often, however, and especially with respect to Russian investment markets, it is necessary to consider statements of problems in order to optimize investment portfolios, which differ fundamentally from the Markowitz scheme. First of all, the fundamental difference between new statements and the Markowitz scheme concerns the need to consider a new definition of investment risk associated with the formation of the investment portfolio.

The thing is that the definition of the calculated values of the efficiency and risks in the Markowitz optimization scheme as the average value of the efficiency and the standard deviation from the mean of efficiency (interpreted as a random variable), respectively, often does not correspond to the real situation on the investment market.

Suppose, for instance, the efficiency of the object of investment has been steadily growing over time. In this case, making a decision about investing in the object in question, we should take not the average value of the predicted values as the efficiency of already embodied values, but the value which is determined by the growth trend for the next period of time designated by the investor as the period of building of the portfolio investments. With this choice of the estimated values of the efficiency as a risk we should take the measure of deviation from the predicted value.

Modeling of processes characterized above will be carried out with the use of models that contain not only current, but also lagged values of factor variables. These models are called *distributed lag models*. Along with the lagged values of factor variables stock returns are influenced by their value in the prior moments of time. Models which contain lagged values of the dependent variable as a factor are called *autoregression model*. We suggest to apply both of these models to the optimization of investment portfolio.

2.1. Consider a model with a distributed lag with respect to the three-factor model (3).

$$D_{iT}=a_i+b_{i10}D_T+b_{i11}D_{T-1}+\ldots+b_{i1k}D_{T-k}+b_{i20}D^I_T+b_{i21}D^I_{T-1}+\ldots+b_{i2k}D^I_{T-k}+$$
$$+b_{i30}D^r_T+b_{i31}D^r_{T-1}+\ldots+b_{i30}D^r_{T-k},i=1,\ldots,n, \qquad (4)$$

According to this model, the stochastic dependence of the i-th shares in the period T on the market of efficiency of the i-th shares in the period T - D_{iT} is carried out not only by the values of the yield of the market in the period T, but also by the values of the yield of the market in prior periods. In addition, the same dynamic dependence is assumed for the efficiencies of an industry and the a region, to which a considered issuing enterprise belongs.

To obtain the coefficients of the model (4), the original information is grouped according to the length of lag, that is its factorization into k classes is made.

An important economic sense have the correlation factors of the resulting model. So, b_{i10} - *the short-term market multiplier*, characterizes the mean absolute change in the efficiency of the i-th shares in the period T with a variation of a unit value of market yield in the period T, excluding the impact of market yield values in prior periods. In the next period the cumulative effect of changes in the value of the market yield on the efficiency of of the i-th shares will be (b_{i10} + b_{i11}), etc. *The long-term market multiplier*

$$b_{i1}=b_{i10}+b_{i11}+\ldots+b_{i1k}$$

shows the absolute change in the efficiency of the i-th shares in the long run under the influence of changes in the a unit value of market return in the current period.

A similar economic sense have a *long-term industry multiplier*–

$$b_{i2}=b_{i20}+b_{i21}+\ldots+b_{i2k}$$

and a *long-term region multiplier* –

$$b_{i3}=b_{i30}+b_{i31}+\ldots+b_{i3k}.$$

2.2. Using the autoregressive model.

Let us apply the autoregressive model to the single-factor Sharpes' model (1).

$$D_{iT}=\alpha_i+\beta_i D_T+c_i D_{iT-1}+\varepsilon_i, \quad i=1,...,n, \qquad (5)$$

As in the distributed lag model, β_i is the short-term industry multiplier of the market, but the long-term industry multiplier is calculated differently. During the period (T + 1) the cumulative effect of the change in the value of the market return on the efficiency of the i-th shares will be βisi in the period (T +2) - $\beta_i c_i^2$, so the long-term the industry multiplier of the market in the autoregressive model, taking into account the conditions of stability ($|c|<1$) is of the following form:

$$b_i=\beta_i+\beta_i c_i+\beta_i c_i^2+\ldots=\frac{\beta_i}{1-c_i}.$$

The autoregressive model is constructed similarly and it uses the three-factor Sharpe's model (3) as the basic.

3. Using the proposed models.

One-factor and three-factor dynamic models in statements (4) and (5) are used in the first stage when it is necessary to determine the yield of of the i-th shares in period T, and this forecast is realized not only through the values of the yield of the overall market in the period T, but also through the market return value of the previous periods, as well as through the dynamics of yields of industry and region, to which belong the shares discussed.

Markowitz model in productions (II_1) and (II_2) is used in the second stage of building a portfolio of assets during the allocation of invested capital in various asset classes: stocks, bonds, real estate, etc. The choice of specific options for the models used in both phases is determined by the presence of initial information, its completeness and accuracy.

Thus, profitability, risk and dynamics in stock returns change - these are the three basic notions that must be considered in the building and managing a portfolio of securities. By combining assets with different rates of return, risk, the relationship between the assets and the dynamics of their return we can build a portfolio with an acceptable ratio of "return / risk", a portfolio with the minimum risk for a given level of return or a portfolio with maximum return with a given level of risk. Accommodation in stocks is one of the most profitable ways to increase ones capital to a private investor This is the very meaning of the portfolio - to find a combination with a satisfactory risk / return ratio.

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About the Author

Yan D. Gelrud

Professor, South Ural State University
Chelyabinsk, Russia

Mr. Yan Gelrud was born in 1947 in Birobidjan (Khabarovsk Territory). In 1965 he finished a school of mathematics and physics at Novosibirsk. In 1970 he graduated from the mathematical faculty of university at Novosibirsk on “Mathematics” speciality. From 1970 to 1991 Yakov was working in the Research Institute of automated control systems as a head of mathematical division. He took part in creation and adoption of more than 100 automated control systems in different branches of industry.

From 1991 to 1997 Mr. Gelrud was doing business, being director general of “URAL-ASCO-SERVICE”. Since the 1st of September 1997 till now he works as a professor of the “Enterprise and management” department in South Ural State University. He teaches a multitude of disciplines, such as “Mathematics”, “Theory of probability and mathematical statistics”, “Econometrics”, “Economic and mathematical methods”, “Mathematical methods of decision-making”, “Bases of decision-making methodology”, “Economical evaluation of investments”, “Mathematical methods and models of project management”, “Studies of managerial systems.”

Yan Gelrud has more than 100 publications and speeches on seminars and conferences of different level. His monograph “Project management in conditions of risk and uncertainty” was published recently. He can be contacted at gelrud@mail.ru